Coherent states of SU(l, 1) groups

1993 J. Phys. A: Math. Gen. 26 7003

(http://iopscience.iop.org/0305-4470/26/23/039)

J. Phys. A: Math. Gen. 26 (1993) 7003–7018. Printed in the UK

Coherent states of $SU(l, 1)$ groups

D M Gitman† and A L Shelepin‡

† Instituto de Física, Departmento de Física Matemática, Universidade de São Paulo, Caixa Postal 20516 01498-970, São Paulo, SP, Brasil

‡ Moscow Institute of Radio Engineering, Electronics and Automation, Prospect Vernadskovo 78, Moscow 117454, Russia

Received 20 May 1993

Abstract. This work can be considered as a continuation of our previous study, in which an explicit form of coherent states (CS) for all $SU(N)$ groups was constructed by means of representations on polynomials. Here we extend that approach to any $SU(l, 1)$ group and construct explicitly corresponding CS. The CS are parametrized by dots of a coset space, which is, in that particular case, the open complex ball CD^l. This space together with the projective space CP^l, which parametrizes the CS of the $SU(l+1)$ group, exhaust all complex spaces of constant curvature. Thus, both sets of CS provide a possibility for an explicit analysis of the quantization problem on all the spaces of constant curvature. This is why the CS of the $SU(N)$ and $SU(l, 1)$ groups are of importance in connection with quantization theory. The constructed CS form an overcompleted system in the representation space and, as quantum states possessing a minimum uncertainty, they minimize an invariant dispersion of the quadratic Casimir operator. The classical limit is investigated in terms of symbols of operators; the limit of the so called star commutator of the symbols generates the Poisson bracket in CD^l, the latter plays the role of the phase space for the corresponding classical mechanics.

1. Introduction

For a long time coherent states (CS) have been widely used in quantum physics [1–5]. On account of the fact that they are parametrized by points of the phase space of a corresponding classical mechanics, they are a natural and convenient tool for establishing a correspondence between classical and quantum descriptions. The CS introduced by Schrödinger and Glauber were mainly used in this context. From a mathematical point of view CS form a continuous basis in Hilbert space (a general description of Hilbert spaces with basis vectors labelled by discrete, continuous, or a mixture of both types of index is given in [6]). As is well known, it is possible to connect quantum mechanical CS with the orbits of Lie groups [7]. In particular, the 'ordinary' CS of Schrödinger and Glauber turned out to be orbits of the Heisenberg–Weyl group. A connection between CS and a quantization of classical systems, in particular systems with a curved phase space, was also established [8]. From this point of view flat phase space corresponds to the Heisenberg–Weyl group and to the Schrödinger–Glauber CS. Kahlerian symplectic manifolds of constant holomorphic curvature can serve as the simplest example of a curved phase space. Such spaces are, for positive curvature, the projective spaces CP^l, and, for negative curvature, the open complex balls CD^l [9]. The groups $SU(N)$, $N = l + 1$ and $SU(l, 1)$ are groups of movements for the spaces CP^l and CD^l correspondingly, and the latter are the coset spaces $SU(N)/U(l)$ and $SU(l, 1)/U(l)$. The quantization on the former is connected with a construction of CS of the groups $SU(N)$, and on the latter with the one of the groups $SU(l, 1)$. These circumstances, besides all other arguments, stress the importance of the investigation of CS for these groups as a first and necessary step in a systematic construction of quantization theory for systems with curved phase spaces. One ought to say the investigation of the CS of these groups has another motivation as well. Their importance for the physics of the group $SU(N)$ is well known and does not need to be explained here. As to the $SU(l,1)$ ones, they often arise in quantum mechanics as groups of dynamical symmetry. For example, the group of the dynamical symmetry of a particle in a magnetic field is $SU(2,1)$ [2], as is the group of dynamical symmetry of Einstein–Maxwell equations for axial-symmetric field configurations [10] and so on.

An explicit form of the CS for any $SU(N)$ group was constructed and investigated in our work [11], using representations of the groups in the space of polynomials of a fixed power. One can also find there references devoted to the CS of the $SU(2)$ group and related questions. In the present work we are going to extend that approach to construct the CS for all $SU(l,1)$ groups. One ought to say that the CS of the $SU(1,1)$ group from that family were first constructed in [7, 12] using the well investigated structure of the $SU(1,1)$ matrices in the fundamental representation. A quantization on the Lobachevsky plane, which is the coset space $SU(1,1)/U(1)$, was considered by Berezin [8, 13], using these CS. It is difficult to use the method from [7, 12] or commutation relations for generators only to construct explicitly CS for any group $SU(l,1)$, since technical complications increase with the number l. Nevertheless, a generalization of the method used by us in [11] allows one to obtain the result, despite the fact that $SU(l,1)$ groups are non-compact and their unitary representations are infinite-dimensional (see the appendix).

We construct CS of the $SU(l,1)$ groups as orbits of highest or lowest weights factorized with respect to stationary subgroups, using representations in spaces of quasi-polynomials of a fixed integer negative power P. The CS are parametrized by points of a coset space, which is, in that particular case, the open complex ball CD^l. As has already been said, this space together with the projective space CP^l, which parametrizes the CS of the $SU(N)$, $N=l+1$, group, exhaust all complex spaces of constant curvature. The constructed CS form an overcompleted system in the representation space and, as quantum states possessing a minimum uncertainty, they minimize an invariant dispersion of the quadratic Casimir operator. The classical limit is investigated in terms of symbols of operators. The role of Planck's constant h is played by $|P|^{-1}$, where P is the signature of the representation. The limit of the so called star commutator of operator symbols generates the Poisson bracket in CD^l, the latter plays the role of the phase space for the corresponding classical mechanics.

In the appendix we add some necessary information about representations of the non-compact groups with which we are working.

2. Construction of CS of $SU(l,1)$ groups

Following the general definition [4, 7] and the method we used for $SU(N)$, we are going to construct the CS of the $SU(l,1)$ groups as orbits in some irreducible representations (IR) of the groups, factorized with respect to stationary subgroups. First, we describe the corresponding representations.

Let g be matrices $N\times N$, $N=l+1$ of a fundamental representation of the group $SU(l,1)$, $g\in SU(l,1)$. They obey the relations

$$\Lambda g^{+}\Lambda=g^{-1}\qquad \det g=1\qquad \Lambda=\begin{pmatrix}1&0\\0&-I_l\end{pmatrix}\qquad \Lambda=\Lambda^{+}=\Lambda^{-1}$$

where I_l is the $l \times l$ unit matrix.

Define by $\mathbb{C}^N$ the N-dimensional space of complex row vectors $z = (z_\mu)$, $\mu = (0, i)$, $i = 1, \ldots, l$, with the scalar product $(z, z')_C = \bar{z}_\mu \Lambda^{\mu\nu} z'_\nu$, and by $\tilde{\mathbb{C}}^N$ the dual space of complex columns $\tilde{z} = (\tilde{z}^\mu)$, with the scalar product $(\tilde{z}, \tilde{z}')_{\tilde{C}} = \bar{\tilde{z}}^\mu \Lambda^{-1}_{\mu\nu} \tilde{z}'^\nu$. The anti-isomorphism of the spaces $\mathbb{C}^N$ and $\tilde{\mathbb{C}}^N$ is given by the relation

$$z \leftrightarrow \tilde{z} \Leftrightarrow z_\mu = \Lambda_{\mu\nu} \bar{\tilde{z}}^\nu \tag{1}$$

on account of the equation $(\tilde{z}, \tilde{z}')_{\tilde{C}} = \overline{(z, z')_C}$. It is convenient to define the mixed Dirac scalar product between elements of $\mathbb{C}^N$ and $\tilde{\mathbb{C}}^N$ as

$$\langle z', \tilde{z} \rangle = (\tilde{z}', \tilde{z})_{\tilde{C}} = \overline{(z', z)_C} = z'_\mu \tilde{z}^\mu. \tag{2}$$

The group acts by its fundamental representations in the spaces $\mathbb{C}^N$ and $\tilde{\mathbb{C}}^N$

$$z_g = zg \qquad \tilde{z}_g = g^{-1}\tilde{z}. \tag{3}$$

The form $\langle z', \tilde{z} \rangle$ is invariant under the group action, $\langle z'_g, \tilde{z}_g \rangle = \langle z', \tilde{z} \rangle$. This means that the whole domain of z_μ can be divided into three invariant subdomains, where $\langle z, \tilde{z} \rangle$ is positive, negative or zero. We restrict ourselves to the subdomain where $\langle z, \tilde{z} \rangle$ is positive, choosing the normalization condition

$$\langle z, \tilde{z} \rangle = |z_0|^2 - \sum_{i=1}^{l} |z_i|^2 = 1 \tag{4}$$

which is sufficient for our purpose of constructing CS connected with the quantization on the coset space CD^l.

Consider spaces Π_P and $\tilde{\Pi}_P$ of quasi-polynomials $\Psi_P(z)$ and $\Psi_P(\tilde{z})$ in z and $\tilde{z}$,

$$\begin{aligned} &\Psi_P(z) = \sum_{\{n\}} K_{\{n\}} \prod_\mu (z_\mu)^{n_\mu} \qquad \Psi_P(z) \in \Pi_P \\ &\Psi_P(\tilde{z}) = \sum_{\{n\}} K_{\{n\}} \prod_\mu (\tilde{z}^\mu)^{n_\mu} \qquad \Psi_P(\tilde{z}) \in \tilde{\Pi}_P \\ &\{n\} = \{n_0, n_1, \ldots, n_l | \sum_\mu n_\mu = P\} \end{aligned} \tag{5}$$

where P are integer and negative, $P < -l$; all n_μ are also integer and $n_0 \leqslant P$, $n_i \geqslant 0$, $i = 1, \ldots, l$.

The fundamental irreducible IR of the group induce unitary IR in the spaces Π_P and $\tilde{\Pi}_P$,

$$\begin{aligned} &T(g)\Psi_P(z) = \Psi_P(z_g) \qquad z_g = zg \qquad \Psi_P \in \Pi_P \\ &\widetilde{T}(g)\Psi_P(\tilde{z}) = \Psi_P(\tilde{z}_g) \qquad \tilde{z}_g = g^{-1}\tilde{z} \qquad \Psi_P(\tilde{z}) \in \widetilde{\Pi}_P. \end{aligned} \tag{6}$$

We will further call P the signature of the IR. Such representations and their place among others of $SU(l, 1)$ groups are described in the appendix.

Define a scalar product of two polynomials from Π_P as

$$\langle\Psi_P|\Psi'_P\rangle = \int \overline{\Psi_P(z)}\Psi'_P(z)\,\mathrm{d}\mu_P(\bar{z},z)$$

$$d\mu_P(\bar{z},z) = \frac{(-P-1)!}{(2\pi)^{l+1}(-P-l-3)!}\delta\left(|z_0|^2 - \sum_{i=1}^{l}|z_i|^2 - 1\right)\prod_{\nu=0}^{l} d\bar{z}_\nu\, dz_\nu \tag{7}$$

$$\mathrm{d}\bar{z}\,\mathrm{d}z = \mathrm{d}(|z|^2)\,\mathrm{d}(\arg z)$$

which can also be interpreted as a mixed Dirac scalar product between elements $|\Psi'_P\rangle = \Psi'_P(z)$ from Π_P and $\langle\Psi_P| = \Psi_P(\bar{z})$ from $\tilde{\Pi}_P$, because of the anti-isomorphism (1).

Note that the restriction to P integer is a way to avoid representations in spaces of multivalued functions; the additional restriction $P < -l$ ensures the existence of the scalar product (7).

The monomials

$$\Psi_{P,\{n\}}(z) = \sqrt{\frac{(-1)^{P-n_0}\Gamma(-n_0)}{n_1!\ldots n_l!\Gamma(-P)}}z_0^{n_0}z_1^{n_1}\ldots z_l^{n_l}$$
$$\{n\} = \left\{n_0, n_1, \ldots, n_l \middle| \sum_\mu n_\mu = P\right\} \tag{8}$$

form a discrete basis in Π_P, whereas the monomials $\Psi_{P,\{n\}}(\bar{z}) = \overline{\Psi_{P,\{n\}}(z)}$ form a basis in $\tilde{\Pi}_P$.

Using the integral

$$\int_1^\infty \mathrm{d}\rho_0 \int_0^\infty \mathrm{d}\rho_1 \ldots \int_0^\infty \mathrm{d}\rho_l\, \delta\left(\rho_0 - \sum_{i=1}^{l}\rho_i - 1\right)\prod_{\nu=0}^{l}\rho_\nu^{n_\nu} = \frac{\prod_{k=1}^{l} n_k!(-\sum_{\mu=0}^{l} n_\mu - l - 3)!}{(-n_0-1)!}$$

it is easy to verify that orthonormality and completeness relations hold:

$$\langle\Psi_{P,\{n\}}|\Psi_{P,\{n'\}}\rangle = \langle P,n|P,n'\rangle = \delta_{\{n\},\{n'\}}$$
$$\sum_{\{n\}} |P,n\rangle\langle P,n| = I_P \tag{9}$$

where I_P is the identity operator in the space of representation of signature P. The monomials (8) obey the remarkable relation

$$\sum_{\{n\}} \Psi_{P,\{n\}}(z')\overline{\Psi_{P,\{n\}}(z)} = \sum_{\{n\}} \Psi_{P,\{n\}}(z')\Psi_{P,\{n\}}(\bar{z}) = \langle z',\bar{z}\rangle^P \tag{10}$$

which is group invariant on account of the invariance of the scalar product (2) under the group transformation, $\langle z'_g, \bar{z}_g\rangle = \langle z', \bar{z}\rangle$. The validity of (10) can be checked by means of the formula

$$(a \pm b)^{-m} = \sum_{n=0}^{\infty}\frac{(m+n-1)!}{(m-1)!n!}a^{-m-n}(\mp b)^n \qquad m > 0 \qquad |b| < |a|$$

together with the binomial formula.

The generators A_ν^μ of the groups $U(l,1) = SU(l,1) \otimes U(1)$ obey the relations (see the appendix)

$$(A_\nu^\mu)^+ = (-1)^{\delta_{\mu 0}+\delta_{\nu 0}} A_\mu^\nu \tag{11}$$

where the Hermitian conjugation is defined with respect to the scalar product (7). Their explicit form in the space Π_P is $A_\mu^\nu = z_\mu \partial/\partial z_\nu$, and in the space $\tilde{\Pi}_P$ is $A_\mu^\nu = \partial/\partial \bar{z}^\mu \bar{z}^\nu$ (action on the left).

Independent generators $\hat{\Gamma}_a$, $a = \overline{1, N^2-1}$, of $SU(l,1)$ can be written through A_μ^ν,

$$\hat{\Gamma}_a = (\Gamma_a)_\mu^\nu A_\nu^\mu \qquad [\hat{\Gamma}_a, \hat{\Gamma}_b] = \mathrm{i} f_{abc} \Gamma_c \tag{12}$$

where Γ_a are generators in a fundamental representation, $[\Gamma_a, \Gamma_b] = \mathrm{i} f_{abc}\Gamma_c$. However, in contrast with the case of $SU(N)$ group, where $\Gamma_a^+ = \Gamma_a$, the Γ_a can be either Hermitian or anti-Hermitian in case of $SU(l,1)$ group. Namely, N matrices Γ_a, with zero diagonal elements and $(\Gamma_a)_\mu^0 = -\overline{(\Gamma_a)_0^\mu}$, differ from the corresponding matrices $SU(N)$ by a factor i only. To be sure, we take those to be the first Γ_a, $a = 1, \ldots, N$. In particular, for $SU(2)$ and $SU(1,1)$ we have

$$\begin{aligned} &SU(2)\colon\ \Gamma_k = \sigma_k \qquad k = 1,2,3 \\ &SU(1,1)\colon\ \Gamma_\lambda = \mathrm{i}\sigma_\lambda \qquad \lambda = 1,2 \qquad \Gamma_3 = \sigma_3 \end{aligned} \tag{13}$$

where the σ_k are the Pauli matrices.

It is easy to verify that condition (11) and the above convention provide the Hermicity of the generators $\hat{\Gamma}_a$.

The quadratic Casimir operator

$$C_2 = \sum_{a=1}^{N^2-1} \epsilon_a \hat{\Gamma}_a^2 \qquad \epsilon_a = \begin{cases} -1 & a = 1, \ldots, N \\ +1 & a = N+1, \ldots, N^2-1 \end{cases} \tag{14}$$

can be written through the A_μ^ν and evaluated explicitly

$$C_2 = \frac{1}{2}\tilde{A}_\mu^\nu \tilde{A}_\nu^\mu = \frac{P(N+P)(N-1)}{2N} \qquad \tilde{A}_\mu^\nu = A_\mu^\nu - \frac{\delta_\mu^\nu}{N}\sum_\lambda A_\lambda^\lambda \tag{15}$$

if one uses the formula

$$\sum_{a=1}^{N^2-1} \epsilon_a (\Gamma_a)_\mu^\nu (\Gamma_a)_\lambda^\alpha = \frac{1}{2}\delta_\lambda^\nu \delta_\mu^\alpha - \frac{1}{2N}\delta_\mu^\nu \delta_\lambda^\alpha$$

which is a generalization to the $SU(l,1)$ group of the well known formula for matrices of the $SU(N)$ group .

Let us construct orbits of a lowest ($D^+(P0)$) or a highest ($D^-(0P)$) weights (of vectors of the basis (8) with the minimal length $\sqrt{\sum_{\mu=0}^{l} n_\mu^2} = |P|$, namely $n_0 = P$, $n_i = 0$. For $D^+(P0)$ the lowest weight is the state $\Psi_{P,\{P0\ldots0\}}(z) = (z_0)^P$. Then we get, in accordance with (6),

$$T(g)\Psi_{P,\{P0\ldots0\}}(z) = \left[z_\mu g_0^\mu\right]^P = (z, \tilde{u})^P \qquad \tilde{u}^\mu = g_0^\mu \tag{16}$$

where the vector $\tilde{u} \in \widetilde{\mathbb{C}}^N$ is the zero column of the $SU(l,1)$ matrix in the fundamental representation.

One can notice that the transformation $\arg \tilde{u}^\mu \to \arg \tilde{u}^\mu + \lambda$ changes all the states (16) by the constant phase $\exp(\mathrm{i}P\lambda)$. To select only physical different quantum states (CS) from all the states of the orbit, one has to impose a gauge condition on $\tilde{u}$, which fixes the total phase of the orbit (16). Such a condition may be chosen in the form $\sum_\mu \arg \tilde{u}^\mu = 0$. Taking into account the fact that the quantities $\tilde{u}$ obey the condition $|\tilde{u}^0|^2 - \sum_{i=1}^l |\tilde{u}^i|^2 = 1$, by definition, as elements of the first column of the $SU(l,1)$ matrix, we get the explicit form of the CS of the $SU(l,1)$ group in the space Π_P:

$$\Psi_{P,\tilde{u}}(z) = \langle z, \tilde{u}\rangle^P \tag{17}$$

$$|\tilde{u}^0|^2 - \sum_{i=1}^{l} |\tilde{u}^i|^2 = 1 \qquad \sum_\mu \arg \tilde{u}^\mu = 0. \tag{18}$$

In the same way we construct the orbit of the highest weight $\Psi_{P,\{P0\ldots0\}}(\tilde{z}) = (\tilde{z}^0)^P$ of $D^-(0P)$ in the space $\tilde{\Pi}_P$, the corresponding CS have the form

$$\Psi_{P,u}(\tilde{z}) = \langle u, \tilde{z}\rangle^P \tag{19}$$

$$|u_0|^2 - \sum_{i=1}^{l} |u_i|^2 = 1 \qquad \sum_\mu \arg u_\mu = 0. \tag{20}$$

One can see that $\Psi_{P,\tilde{u}}(z) = \overline{\Psi_{P,u}(\tilde{z})}$, $z \leftrightarrow \tilde{z}$, $u \leftrightarrow \tilde{u}$.

The quantities $\tilde{u}$ and u, which parametrize the CS (17) and (19), are elements of the coset space $SU(l,1)/U(l)$, in accordance with the fact that the stationary subgroups of both the initial vectors from the spaces Π_P and $\tilde{\Pi}_P$ are $U(l)$. At the same time, the coset space is the l-dimensional open complex ball CD^l of unit radius. Equations (18) or (20) are just possible conditions which define the space. The coordinates u or $\tilde{u}$ are called homogeneous in the CD^l. One can also introduce local independent coordinates α_i, $i = 1, \ldots, l$, $\sum_{i=1}^l |\alpha_i|^2 < 1$ on CD^l. For instance, in the domain where $u_0 \neq 0$, the local coordinates are

$$\alpha_i = u_i/u_0$$
$$u_i = \alpha_i u_0 \qquad u_0 = \frac{\exp(-(\mathrm{i}/N)\sum_{k=1}^l \arg \alpha_k)}{\sqrt{1 - \sum_{k=1}^l |\alpha_k|^2}}. \tag{21}$$

To decompose the CS in the discrete basis one can use the relation (10), since the right-hand side of equation (10) can be treated as CS (17) or (19),

$$\Psi_{P,\tilde{u}}(z) = \sum_{\{n\}} \Psi_{P,\{n\}}(\tilde{u})\Psi_{P,\{n\}}(z). \tag{22}$$

Using Dirac's notations, we get

$$\langle P, u|P, n\rangle = \Psi_{P,\{n\}}(u) \qquad \langle P, n|P, u\rangle = \Psi_{P,\{n\}}(\tilde{u}). \tag{23}$$

Thus, the discrete bases in the spaces Π_P and $\tilde{\Pi}_P$ are the ones in the CS representation.

The completeness relation can be derived similarly to the case of the $SU(N)$ groups [11],

$$\int |P, u\rangle\langle P, u| \mathrm{d}\mu_P(\tilde{u}, u) = I_P. \tag{24}$$

3. Uncertainty relation and CS overlap

The elements of the orbit of each vector of the discrete basis $|P, n\rangle$ and, in particular, the constructed CS are eigenstates for a nonlinear operator C_2', which is defined by its action on an arbitrary vector $|\Psi\rangle$ as

$$C_2'|\Psi\rangle = \sum_a \epsilon_a \langle\Psi|\hat{\Gamma}_a|\Psi\rangle\hat{\Gamma}_a|\Psi\rangle$$

with ϵ_a from (14). The proof of this fact is fully analogous to the one for the $SU(N)$ group [11]. Direct calculations result in

$$C_2'|P, n\rangle = \lambda(P, n)|P, n\rangle$$
$$\lambda(P, n) = \frac{1}{2}\left(\sum_\mu n_\mu^2 - P^2/N\right) = \frac{1}{2}\sum_\mu (n_\mu - P/N)^2. \qquad (25)$$

The eigenvalue $\lambda(P, n)$ attains its minimum for the lowest weight $(D^+(P0))$, for which $\sum_\mu n_\mu^2 = P^2 = \min$. The CS $|P, u\rangle$ belong to the orbit of the lowest weight $\{n\} = \{P0\ldots0\}$. Thus, we get

$$C_2'|P, u\rangle = \frac{P^2(N-1)}{2N}|P, u\rangle. \qquad (26)$$

Define a dispersion of the square of the 'hyperbolic length' of the isospin vector,

$$\Delta C_2 = \langle\Psi|\sum_a \epsilon_a \hat{\Gamma}_a^2|\Psi\rangle - \sum_a \epsilon_a \langle\Psi|\hat{\Gamma}_a|\Psi\rangle^2 = \langle\Psi|C_2 - C_2'|\Psi\rangle$$

where C_2 is a quadratic Casimir operator (14). The dispersion serves as a measure of the uncertainty of the state $|\Psi\rangle$. Due to the properties of the operators C_2 and C_2', it is group invariant and its modulus attains its lowest value $P(N-1)/2$ for the orbits of lowest $(D^+(P0))$ or highest $(D^-(0P))$ weights, particularly for the constructed CS, compared with all the orbits of the discrete basis (8). The relative dispersion of the square of the 'hyperbolic length' of the isospin vector has the value in the CS

$$\frac{\Delta C_2}{C_2} = \frac{N}{N+P} \qquad P < -N - 1 \qquad (27)$$

and tends to zero with $h \to 0$, $h = 1/|P|$. Note, that the relative dispersion here obeys the relation $-\infty < \Delta C_2/C_2 < 0$, in contrast to the case of compact groups $SU(N)$, where $0 < \Delta C_2/C_2 \leqslant 1$.

Proceeding to the CS overlap, one has to say that many of its properties were generally investigated in [7, 15–17]. Using the completeness relation (9) and formulae (23), (10) and (17), we get for the overlap of the CS in question

$$\begin{aligned}\langle P, u|P, v\rangle &= \sum_{\{n\}} \langle P, u|P, n\rangle\langle P, n|P, v\rangle \\ &= \sum_{\{n\}} \Psi_{P,\{n\}}(u)\Psi_{P,\{n\}}(\tilde{v}) \\ &= (u, \tilde{v})^P = \Psi_{P,\tilde{v}}(u).\end{aligned} \qquad (28)$$

As in the case of the Heisenberg–Weyl and $SU(N)$ groups, the CS overlap plays here the role of the δ-function (the so called reproducing kernel). Namely, if $\Psi_P(u)$ is a vector $|\Psi\rangle$ in the CS representation, $\Psi_P(u) = \langle P, u|\Psi\rangle$, then

$$\Psi_P(u) = \int \langle P, u|P, v\rangle \Psi_P(v)\, \mathrm{d}\mu_P(\bar{v}, v).$$

The modulus of the CS overlap (28) has the following properties:

$$\begin{aligned} &|\langle P, u|P, v\rangle| \leqslant 1 \qquad \lim_{P\to\infty} |\langle P, u|P, v\rangle = 0 \qquad \text{if } u \neq v \\ &|\langle P, u|P, v\rangle| = 1 \qquad \text{only if } u = v \end{aligned} \tag{29}$$

which allows a symmetric† $s(u, v)$ to be introduced in CD^l,

$$s^2(u, v) = -\ln|\langle P, u|P, v\rangle|^2 = -P\ln|(u, \bar{v})|^2. \tag{30}$$

The symmetric $s(u, v)$ generates the metric tensor in the space CD^l. To demonstrate that, it is convenient to go over to local independent coordinates (21). In local coordinates the symmetric takes the form

$$s^2(\alpha, \beta) = -P \ln \frac{\lambda(\alpha, \bar{\beta})\lambda(\beta, \bar{\alpha})}{\lambda(\alpha, \bar{\alpha})\lambda(\beta, \bar{\beta})} \tag{31}$$

with $\lambda(\alpha, \bar{\beta}) = 1 - \sum_i \alpha_i \bar{\beta}_i$. Calculating the square of the 'distance' between two infinitesimally close points α and $\alpha + \mathrm{d}\alpha$, one finds

$$\begin{aligned} &\mathrm{d}s^2 = g_{i\bar{k}}\mathrm{d}\alpha_i \mathrm{d}\bar{\alpha}_k \qquad g_{i\bar{k}} = -P\lambda^{-2}(\alpha, \bar{\alpha})[\lambda(\alpha, \bar{\alpha})\delta_{ik} + \bar{\alpha}_i\alpha_k] \\ &g_{i\bar{k}} = \frac{\partial^2 F}{\partial\alpha_i \partial\bar{\alpha}_k} \qquad F = P\ln\lambda(\alpha, \bar{\alpha}) \\ &\det\|g_{i\bar{k}}\| = P^l\lambda^{-N}(\alpha, \bar{\alpha}) \qquad g^{\bar{k}i} = -\frac{1}{P}\lambda(\alpha, \bar{\alpha})(\delta_{ki} - \bar{\alpha}_k\alpha_i). \end{aligned} \tag{32}$$

The quantity $g_{i\bar{k}}$ is the metric on the open complex ball CD^l with constant holomorphic sectional curvature $C = 2/P < 0$ [9], whereas $g^{\bar{k}i}$ defines the corresponding Poisson bracket on this Kahlerian manifold

$$\{f, g\} = \mathrm{i}g^{\bar{k}i}\left(\frac{\partial f}{\partial\alpha_i}\frac{\partial g}{\partial\bar{\alpha}_k} - \frac{\partial f}{\partial\bar{\alpha}_k}\frac{\partial g}{\partial\alpha_i}\right). \tag{33}$$

As we have just said, the logarithm of the modulus of CS overlap defines a symmetric on the coset space. The expression for the symmetric through CS has one and the same form for any group; its existence follows directly from the properties of CS. As for the real distance ρ on the coset space, its expression through CS depends on the group. For example, in case of the CP^l (the $SU(l+1)$ group), $\cos(\rho/P) = |(u, \bar{v})|$, so that for $l = 1$, ρ is the distance on the sphere with radius $P/2$. For our case of CD^l (the $SU(l, 1)$ group) the distance ρ shows up in the relation $\cosh(\rho/P) = |(u, \bar{v})|$. Thus, for both cases (see [11] as well) we have the following relations between CS overlaps and the distances

$$\begin{aligned} &CP^l\colon\ |\langle P, u|P, v'\rangle| = [\cos(\rho/P)]^P \\ &CD^l\colon\ |\langle P, u|P, v'\rangle| = [\cosh(\rho/P)]^P. \end{aligned} \tag{34}$$

† We remember that a real and positive symmetric obeys only two axioms of a distance ($s(u, v) = s(v, u)$ and $s(u, v) = 0$, if and only if $u = v$), except the triangle axiom.

4. Operators symbols and classical limit

We are going to investigate the classical limit on the language of operator symbols, constructed by means of CS. Remember that the covariant symbol $Q_A(u,\bar{u})$ and the contravariant one $P_A(u,\bar{u})$ of an operator $\hat{A}$ are defined as [13, 14]

$$
\begin{aligned}
&Q_A(u,\bar{u}) = \langle P,u|\hat{A}|P,u\rangle \qquad \hat{A} = \int P_A(u,\bar{u})|P,u\langle\rangle P,u|\mathrm{d}\mu_P(\bar{u},u) \\
&Q_A(u,\bar{u}) = \int P_A(u,\bar{u})|\langle P,u|P,v\rangle|^2 \mathrm{d}\mu_P(\bar{u},u).
\end{aligned} \tag{35}
$$

One can calculate the P and Q symbols of operators explicitly, if one formally generalizes the creation and annihilation operator method to the case under investigation. Consider for example IR $D^+(P0)$ and introduce, as in $SU(N)$, operators $a_\mu^\dagger$ and a^ν, which act on basis vectors and CS by the formulae

$$
\begin{aligned}
a_\mu^\dagger|P,n\rangle &= \sqrt{\frac{n_\mu+1}{P+1}}|P+1,\ldots,n_\mu+1,\ldots\rangle = z_\mu\Psi_{P,\{n\}}(z) \\
a^\mu|P,n\rangle &= \sqrt{Pn_\mu}|P-1,\ldots,n_\mu-1,\ldots\rangle = \frac{\partial}{\partial z_\mu}\Psi_{P,\{n\}}(z) \\
\langle P,n|a_\mu^\dagger &= \sqrt{\frac{n_\mu}{P}}\langle P-1,\ldots,n_\mu-1,\ldots| = \frac{1}{P}\frac{\partial}{\partial\bar{z}^\mu}\Psi_{P,\{n\}}(\bar{z}) \\
\langle P,n|a^\mu &= \sqrt{(P+1)(n_\mu+1)}\langle P+1,\ldots,n_\mu+1,\ldots| \\
&= (P+1)\bar{z}^\mu\Psi_{P,\{n\}}(\bar{z}) \\
a^\mu|P,u\rangle &= P\bar{u}^\mu|P-1,u\rangle = \frac{\partial}{\partial z_\mu}\Psi_{P,\bar{u}}(z) \\
\langle P,u|a_\mu^\dagger &= u_\mu\langle P-1,u| = \frac{1}{P}\frac{\partial}{\partial\bar{z}^\mu}\Psi_{P,u}(\bar{z}) \\
[a^\mu,a_\nu^\dagger] &= \delta_\nu^\mu \qquad [a^\mu,a^\nu] = [a_\mu^\dagger,a_\nu^\dagger] = 0.
\end{aligned} \tag{36}
$$

(Note that the sign † does not mean the Hermitian conjugation with respect to the scalar product (7).) In contrast to the $SU(N)$ group where P and n_μ are always positive, P and n_0 are negative for the $SU(l,1)$ group, so that complex factors can appear when the operators $a_\mu^\dagger$ and a^μ act on states. Because of negative n_0, the space of states cannot be treated as a Fock space.

Quadratic combinations $A_\mu^\nu = a_\mu^\dagger a^\nu = z_\mu\partial/\partial z_\nu$ obey commutation relations (42) and are generators of the groups $U(l,1) = SU(l,1)\otimes U(1)$. This is why operators which are polynomial in the generators can be written using $a_\mu^\dagger$ and a^ν and presented in the normal or anti-normal form,

$$
\hat{A} = \sum_K A^{\mu_1\ldots\mu_K}_{\nu_1\ldots\nu_K} a_{\mu_1}^\dagger\ldots a_{\mu_K}^\dagger a^{\nu_1}\ldots a^{\nu_K} = \sum_K \tilde{A}^{\mu_1\ldots\mu_K}_{\nu_1\ldots\nu_K} a^{\nu_1}\ldots a^{\nu_K} a_{\mu_1}^\dagger\ldots a_{\mu_K}^\dagger. \tag{37}
$$

Direct calculations give for the symbols of such operators:

$$Q_A(u,\bar{u}) = \sum_K (-1)^{K-K_0} \frac{(-P+K-1)!}{(-P-1)!} A^{\mu_1\ldots\mu_K}_{\nu_1\ldots\nu_K} u_{\mu_1}\ldots u_{\mu_K}\bar{u}_{\nu_1}\ldots\bar{u}_{\nu_K}$$

$$P_A(u,\bar{u}) = \sum_K (-1)^{K-K_0} \frac{(-P-N-K)!}{(-P-N)!} \tilde{A}^{\mu_1\ldots\mu_K}_{\nu_1\ldots\nu_K} u_{\mu_1}\ldots u_{\mu_K}\bar{u}_{\nu_1}\ldots\bar{u}_{\nu_K} \qquad (38)$$

$$K_0 = \sum_{i=1}^{l} \delta_{\nu_i,0}.$$

In manipulations it is convenient to deal with non-diagonal symbols

$$Q_A(u,\bar{v}) = \frac{\langle P,u|\hat{A}|P,v\rangle}{\langle P,u|P,v\rangle}$$

which can be derived from the corresponding diagonal symbols (38) by the replacement $\bar{u}\to\bar{v}$ and by multiplying each term by the factor $\langle u,\bar{v}\rangle$. In the local independent variables (21) these symbols are analytical functions of both their arguments.

Consider for example covariant symbols $\langle\hat{J}_a\rangle = \langle P,u|\hat{J}_a|P,u\rangle$ of generators $\hat{J}_a = (\Gamma_a)^\nu_\mu A^\mu_\nu$ for the $SU(1,1)$ group, so that Γ_a are matrices (13)). In this case it is convenient to parameterize the CS by j,θ,φ; $P/2=j$, $\bar{u}^1 = \cosh\frac{1}{2}\theta e^{-i\varphi}$, $\bar{u}^2 = \sinh\frac{1}{2}\theta e^{-i\varphi}$,

$$\begin{aligned} \langle\hat{J}_1\rangle &= \mp j\sinh\theta\cos\varphi = j_1 \\ \langle\hat{J}_2\rangle &= \mp j\sinh\theta\sin\varphi = j_2 \\ \langle\hat{J}_3\rangle &= \mp j\cosh\theta = j_3 \qquad -j_1^2-j_2^2+j_3^2=j^2 \end{aligned} \qquad (39)$$

where the upper sign belongs to $D^+(P)$ and lower one to $D^-(P)$.

The dots on the axis $\langle\hat{J}_3\rangle$ correspond to the states of discrete basis $|j,m\rangle$, $\hat{J}_3|j,m\rangle = m|j,m\rangle$; the CS are placed on the upper $(D^+(P0))$ or lower $(D^-(0P))$ sheet of the two sheet hyperboloid on figure 1.

The classical limit can be considered as in [11]. So, one can obtain for the star product of two covariant symbols in the local coordinates (21) the following expression

$$\begin{aligned} Q_{A_1}\star Q_{A_2} &= Q_{A_1A_2}(\alpha,\bar{\alpha}) = \int Q_{A_1}(\alpha,\bar{\beta})Q_{A_2}(\beta,\bar{\alpha})e^{-s^2(\alpha,\beta)}\,d\mu_P(\bar{\beta},\beta) \\ &= Q_{A_1}(\alpha,\bar{\alpha})Q_{A_2}(\alpha,\bar{\alpha}) + g^{i\bar{k}}\frac{\partial Q_{A_1}(\alpha,\bar{\alpha})}{\partial\bar{\alpha}_k}\frac{\partial Q_{A_2}(\alpha,\bar{\alpha})}{\partial\alpha_i} + o(h) \end{aligned} \qquad (40)$$

$$d\mu_P(\bar{\beta},\beta) = \frac{(-P-1)!}{(-P-l-3)!P^l}\det\|g_{l\bar{m}}(\beta,\bar{\beta})\|\prod_{i=1}^{l}\frac{d\,\mathrm{Re}\,\beta_i\,d\,\mathrm{Im}\,\beta_i}{\pi}$$

where the matrix $g^{i\bar{k}}$ was defined in (32) and is proportional to $h=1/|P|$.

Note, the decomposition of $Q_{A_1}(\alpha,\bar{\beta})Q_{A_2}(\beta,\bar{\alpha})$ into a series with respect to $\beta-\alpha$ is possible if symbols are non-singular (differentiable) functions on $\alpha,\bar{\beta}$ in the limit $P\to\infty$. This is valid for polynomial operators, but not for the operators of finite transformations, which are singular in that limit.

Taking expression (33) for the Poisson bracket in the space CD^l, and equation (40) into account we get for the star multiplication of two symbols of polynomial operators

$$\begin{aligned} &\lim_{h\to 0} Q_{A_1}\star Q_{A_2} = Q_{A_1}Q_{A_2} \\ &Q_{A_1}\star Q_{A_2} - Q_{A_2}\star Q_{A_1} = i\{Q_{A_1},Q_{A_2}\} + o(h). \end{aligned} \qquad (41)$$

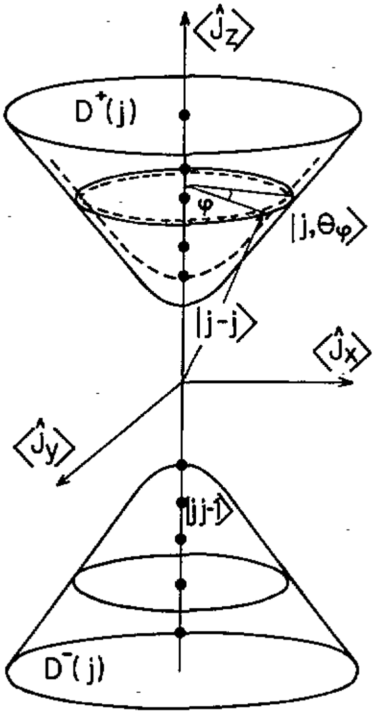

Figure 1. Mean values in cs of the $SU(1,1)$ generators.

Equations (41) are just Berezin's conditions for the classical limit in terms of operator symbols [7, 13], where the quantity $h = 1/|P|$ plays the role of the Planck constant. This property of h has been remarked upon already in section 3, while investigating the uncertainty relation. From that consideration it is also easy to see that the length of the isospin vector is proportional to the signature P of a representation. Thus, the classical limit in this case is connected with large values of the isospin vector. In contrast to the ordinary case of the Heisenberg–Weyl group, where the Planck constant is fixed, as for $SU(N)$, the Planck constant can really take different values, which are, however, quantized since the quantity P is discrete.

It is easy to demonstrate that the contravariant and covariant symbols coincide in the classical limit. For instance,

$$Q_A(\alpha, \bar{\alpha}) = P_A(\alpha, \bar{\alpha}) + g^{ik} \frac{\partial P_A(\alpha, \bar{\alpha})}{\partial \bar{\alpha}_k \partial \alpha_i} + \mathrm{o}(h).$$

For the operators of finite transformations one can derive

$$Q_{T(g_2)T(g_1)}(u, \bar{u}) = \langle P, u | T(g_2) T(g_1) | P, u \rangle = (u, g_2 g_1 \tilde{v})^P$$

$$Q_{T(g_2)} \star Q_{T(g_1)} = Q_{T(g_2 g_1)}.$$

We see that the law of multiplication of these symbols is similar to that of matrices of finite transformations and does not depend on P. Thus, we have an example of operators, which do not obey equation (39) in classical physics. According to Yaffe's terminology [15] these are the so called non-classical operators.

5. Conclusion

Thus, an explicit construction of the CS for all the $SU(l,1)$ groups appears to be possible as well as for all $SU(N)$ ones due to an appropriate choice for the irreducible representations of the group in the space of polynomials and quasi-polynomials of a fixed power. Many formulae look very similar in the two cases, nevertheless there are also many differences connected with the principal difference between the compact $SU(N)$ and non-compact $SU(l,1)$. Construction of the CS of the two groups provides an explicit analysis of the quantization problem on complex spaces of constant curvature in full agreement with the general theory [17] of quantization on Kahlerian manifolds.

Appendix

We give here a brief description of discrete positive D^+ and negative D^- series of unitary IR of $SU(l,m)$, in particular, $SU(l,1)$ ones, which are related to the CS in question.

Remember first, if r is a rank of a semi-simple algebra Lie, which it is our case, then there exist r fundamental IR $D_1,\ldots,D_r$, having the corresponding highest weights $M_1,\ldots,M_r$. Consider the tensor product of the representations

$$D_1^{P_1}\otimes D_2^{P_2}\ldots D_r^{P_r}$$

where P_i are non-negative integers, and $D_i^{P_i}$ means the P_i multiplied by the direct product of the D_i. Let $D(P_1,\ldots,P_r)$ be the irreducible part of this product, containing the highest weight $M(P)=\sum P_iM_i$, then all finite-dimensional IR (and therefore all unitary IR of compact groups) are exhausted by such representations. The set of numbers $P_1,\ldots,P_r$ is called the signature of IR. Fundamental IR are characterized by one non-zero index of signature, which is unity. For unitary IR of non-compact groups one needs to consider, in general, complex P_i, i.e. to generalize the tensor calculus and consider tensors of non-integer or complex ranks [18]. In contrast to the case of compact groups, all linear unitary IR of non-compact groups are infinite-dimensional. In this case there are two different types of representation space, which correspond to discrete and continuous series. The theory of discrete series is mostly analogous to the finite-dimensional case. In [8–24] a classification of unitary IR of $SU(l,m)$. The case of $SU(l,1)$ is considered separately in [25]; besides, one can find the case of $SU(2,1)$ in [26] and the case of $SU(2,2)$ in [27].

The fundamental IR $D(10\ldots0)$ and $D(0\ldots0)$ of $SU(l,m)$ groups are representations by $N\otimes N$, $N=l+m$, quasi-unimodular matrices g and g^{-1}, $\Lambda g^+\Lambda=g^{-1}$, $\Lambda=\mathrm{diag}_{l,m}(1,\ldots,1,-1,\ldots,-1)$ in spaces of N-dimensional rows z_μ or columns $\bar{z}^\mu$, see e.g. (3). Other fundamental IR $D(010\ldots0), D(001\ldots0),\ldots,D(0\ldots100), D(0\ldots010)$ are realized in spaces of antisymmetric elements $z_{ik}, z_{ikm},\ldots,\bar{z}^{ikm},\bar{z}^{ik}$, [18, 19, 26, 30].

As is well known the commutation relations of $U(l,m)$ generators have the form

$$[A_\mu^\nu, A_\lambda^k]=\delta_\lambda^\nu A_\mu^k-\delta_\mu^k A_\lambda^\nu \tag{42}$$

and furthermore, for unitary IR [19, 21]

$$(A_j^k)^+=\epsilon_k^j A_k^j \qquad \epsilon_k^j=\begin{cases}+1 & \text{if } k,j\leqslant l \text{ or } k,j>l\\ -1 & \text{if } k\leqslant l<j \text{ or } j\leqslant l<k.\end{cases} \tag{43}$$

It is convenient to introduce a basis, consisting of eigenfunctions of the commuting operators A_i^i,

$$A_i^i|n_1 n_2 \dots n_N\rangle = n_i|n_1 n_2 \dots n_N\rangle \qquad N = l + m$$

where the n_i are called occupation numbers. By means of the commutation relations (42) we get for $i \neq k$

$$A_k^i|\dots n_i \dots n_k \dots\rangle = \sqrt{n_i(n_k+1)}|\dots n_i - 1 \dots n_k + 1 \dots\rangle. \tag{44}$$

The conditions

$$\begin{aligned} &n_k(n_j+1) \geqslant 0 \qquad k, j \leqslant l \text{ or } k, j > l \\ &n_k(n_j+1) \leqslant 0 \qquad k \leqslant l < j \text{ or } j \leqslant l < k \end{aligned} \tag{45}$$

must hold for unitary IR.

One can reach any weight of a given IR by means of operators A_k^i, moving from any other weight of the representation; the weight diagram stops suddenly when one reaches a highest weight, the factor in (44) appears to be zero at this step. The occupation number space is N-dimensional; weights, which correspond to a given IR, fill in a area with $\sum n_i = P$, where P is an eigenvalue of the operator $\sum A_i^i$, commuting with all the operators A_k^i.

Consider some particular cases. For the groups $SU(2,1)$ and $SU(3)$ the weights fill in the three-dimensional space (figure A1(a)); the weights which correspond to one IR fill in areas on the planes $n_1 + n_2 + n_3 = P$ (such areas for integer n_i are represented on figure A1(b)).

For unitary IR of $SU(3)$ either $n_i \geqslant 0$ or $n_i \leqslant -1$ and are integers. Unitary IR are finite-dimensional; areas with $P \geqslant 0$ correspond to IR $D^0(P0)$, ones with $P \leqslant -3$ correspond to $D^0(0Q)$, $Q = -P - 3 = \sum_i q_i$, $q_i = -p_i - 1 \geqslant 0$. The representations $D^0(P0)$ and $D^0(0P)$ are conjugated. One can find the following unitary IR for $SU(2,1)$, using (44) and (45): bounded below by the weight with $Y_{\min}$, $Y = -P/3 - n_3$,

$$D^+(P0),\ P < 0,\ n_1, n_2 \geqslant 0 \text{ and integers, } n_3 \leqslant 0 \text{ and real}$$

$$D_1^+(P0),\ P \geqslant 0,\ n_1, n_2 \geqslant 0 \text{ and integers, } n_3 \leqslant -1 \text{ and integer}$$

and IR conjugated to the former, bounded above by weights with $Y_{\max}$, $Y = -Q/3 + q_3$,

$$D^-(0Q),\ Q < 0,\ q_1, q_2 \geqslant 0 \text{ and integers, } q_3 \leqslant 0 \text{ and real}$$

$$D_1^-(0Q),\ Q \geqslant 0,\ q_1, q_2 \geqslant 0 \text{ and integers, } q_3 \leqslant -1 \text{ and integer}$$

$$Q = -P - 3,\ q_i = -n_i - 1.$$

The replacement of the signature of IR $D(P0) \to D(0 - P - 3)$ is a particular case of the group of parameters transpositions of IR [19]. Such replacements leave eigenvalues of the Casimir operator unchanged.

Weights of IR for the $SU(4)$, $SU(3,1)$, $SU(2,2)$ groups fill in areas in the space $n_1 + n_2 + n_3 + n_4 = P$; such areas, for n_i integers, are shown on figure A1(c).

The unitary IR $D(P0\dots0)$ and $D(0\dots0P)$ of the $SU(N)$ are well known full symmetrical representations. They can be realized, for instance, in spaces of polynomials of a fixed power P.

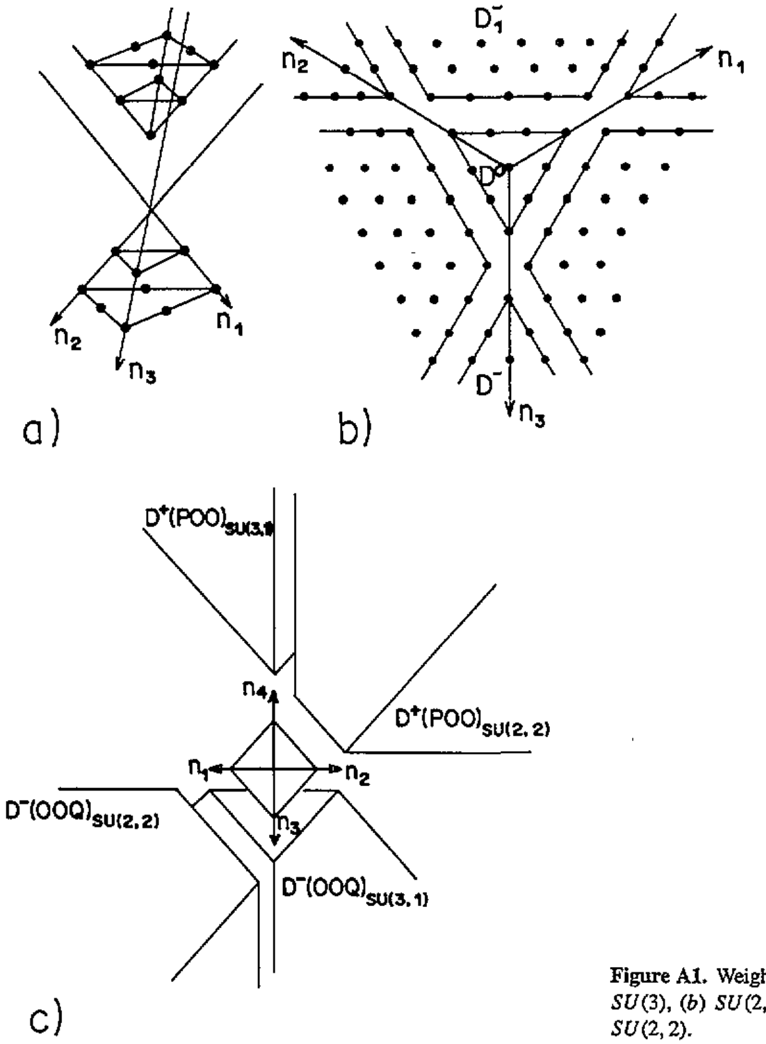

Figure A1. Weight diagrams for groups (*a*) $SU(3)$, (*b*) $SU(2,1)$, (*c*) $SU(4)$, $SU(3,1)$, $SU(2,2)$.

Weights diagrams of the unitary IR, corresponding to the discrete series $D^+(P0\ldots0)$ and $D^-(0\ldots0P)$ of the $SU(l,1)$ groups are presented on the figure A2.

They fall in a sum of full symmetrical IR by the reduction on the compact subgroup $SU(l)$

$$D^+(P0\ldots0)_{SU(l,1)} = \sum_{\alpha=0}^{\infty} D(\alpha0\ldots0)_{SU(l)}$$

$$D^-(0\ldots0P)_{SU(l,1)} = \sum_{\alpha=0}^{\infty} D(0\ldots0\alpha)_{SU(l)}.$$

This is easily seen on figure A2: each level of the weight diagram corresponds to an IR of a subgroup. Besides the eigenvalues of the $l-1$ commuting Cartan generators H_l from the compact subgroup $SU(l)$ (these are linear combinations of A^i_i, $i \neq 0$), the weights of

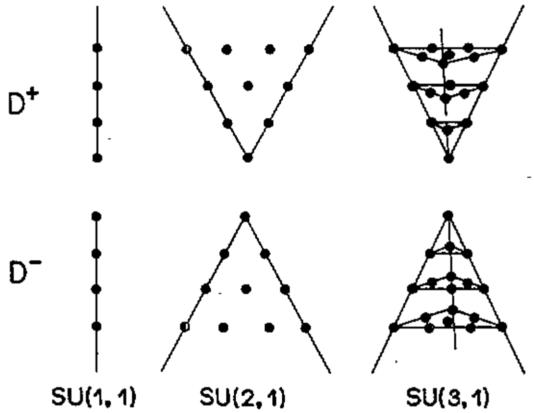

Figure A2. Weight diagrams of discrete series D^+ and D^- for groups $SU(1,1)$, $SU(2,1)$, $SU(3,1)$.

$SU(l, 1)$ are characterized by an additional number $Y = \mp(P/N - n_0)$; the upper sign for $D^+(P0\ldots0)$, the lower one for $D^-(0\ldots0P)$), and weight diagrams are bounded below and above correspondingly,

$$Y_{\min} = -P(N-1)/N \qquad Y_{\max} = P(N-1)/N. \tag{46}$$

The weight structure of these IR does not depend on P; only the position of the diagram in the weight space depends on P, according to (46).

References

[1] Klauder J R and Sudarshan E C 1968 *Fundamentals of Quantum Optics* (New York: Benjamin)
[2] Malkin I A and Man'ko V I 1979 *Dynamical Symmetries and Coherent States of Quantum Systems* (Moscow: Nauka) p 320
[3] Klauder I R and Skagerstam B S 1985 *Coherent States, Applications in Physics and Mathematical Physics* (Singapore: World Scientific) p 991
[4] Perelomov A M 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
[5] Bagrov V G and Gitman D M 1990 *Exact Solutions of Relativistic Wave Equations* (Dordrecht: Kluwer) p 323
[6] Twareque Ali S, Antoine J-P and Gazeau J-P 1993 *Ann. Phys.* **222** 1, 38
[7] Perelomov A M 1972 *Commun. Math. Phys.* **26** 222
[8] Berezin F A 1975 *Commun. Math. Phys.* **40** 153
[9] Kobayashi S and Nomizu K 1969 *Foundation of Differential Geometry* vols 1, 2 (New York: Interscience)
[10] Kinnersley M 1972 *J. Math. Phys.* **14** 651
[11] Gitman D M and Shelepin A L 1993 *J. Phys. A: Math. Gen.* **26** 313; *Proc. XVIII Int. Coll. on Group Theoretical Methods in Physics (Moscow, June 1990)* (New York: Nova Science) p 251
Gitman D M, Kharchev S M and Shelepin A L 1990 *Trudy PhIAN (Proc. Lebedev Inst.)* **201** 95
[12] Monastyrsky M I and Perelomov A M 1972 *Dokl. Acad. Nauk USSR* **207** 1303
[13] Berezin F A 1986 *The Method of Second Quantization* (Moscow: Nauka)
[14] Berezin F A 1980 *Usp. Fiz. Nauk* **132** 497
Berezin F A and Shubin M A 1983 *Schrödinger Equation* (Moscow: Moscow State University)
[15] Yaffe L G 1982 *Rev. Mod. Phys.* **54** 407
[16] Smorodinski Ya A, Shelepin A L and Shelepin L A 1992 *Usp. Fiz. Nauk* **162** 12, 1
[17] Berezin F A 1975 *Izv. Acad. Nauk USSR, Ser. Math.* **39** 363
[18] Fronsdal C 1965 *High Energy Physics and Elementary Particles* (Vienna: IAEA) p 585
[19] Shelepin A L and Shelepin L A 1989 *Trudy PhIAN (Proc. Lebedev Inst.)* **191** 46
[20] Bargmann V 1947 *Ann. Math.* **48** 568
[21] Barut A O and Raczka R 1977 *Theory of Group Representations and Applications* (Warsaw: PWN)

[22] Gelfand I M and Graev M I 1965 *Izv. Acad. Nauk USSR, Ser. Math.* **29** 1329
[23] Joseph D W 1970 *J. Math. Phys.* **11** 1279
[24] Vilenkin N Ya and Shapiro R L 1980 *Group Theoretical Methods in Physics* vol 1 (Moscow: Nauka) p 4
[25] Ottoson U 1968 *Commun. Math. Phys.* **10** 114
[26] Fronsdal C 1965 *Proc. R. Soc.* A **288** 98
[27] Tsu Yao 1967 *J. Math. Phys.* **8** 1931
[28] Arecchi F T, Courtens E, Gilmore R and Thomas H 1972 *Phys. Rev.* **6** 2211
[29] Lieb E M 1973 *Commun. Math. Phys.* **31** 327
[30] Shelepin L A 1973 *Trudy PhIAN (Proc. Lebedev Inst.)* **70** 3
[31] Barut A O and Fronsdal C 1965 *Proc. R. Soc.* A **287** 532